Analytical and numerical approaches to structure functions in magnetohydrodynamic turbulence

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Abstract

In magnetohydrodynamic turbulence, the classical theory by Kraichnan and Iroshnikov based on dimensional analysis gives a linear dependence of the exponents $\zeta_p = p/4$ of the structure functions for the Elsässer variables $\mathbf{z}^{\pm} = \mathbf{u} \pm \mathbf{B}$. This linear behavior contradicts observations of MHD turbulence in the solar wind, where anomalous scaling was found similar as in hydrodynamic turbulence. Since the experimentally observed scaling can not yet be derived by analytical theories, one is dependent also on numerical simulations. As an alternative to direct numerical simulations we present a stochastic approach that recently was introduced for two-dimensional hydrodynamic flows. Finally, we discuss the applicability of operator-product expansions on a direct cascade in strongly turbulent systems.

1 Introduction

Magnetohydrodynamic (MHD) turbulence offers a possibility to study strongly turbulent systems even in two dimensions. This is in contrast to two-dimensional hydrodynamic turbulence where the conservation of enstrophy suppresses small scale intermittency. In addition, the dimensionality in the MHD equations is believed to play only a minor role in the turbulent behavior. Therefore, the two-dimensionality makes it possible to achieve a reasonably resolved inertial range in numerical simulations which is not yet possible for the three-dimensional problems.

The incompressible magnetohydrodynamic equations read

$$\partial_t \mathbf{z}^{\pm} + \mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} + \nabla p = \nu \Delta \mathbf{z}^{\pm} \ , \ \ \mathrm{div}\ \mathbf{z}^{\pm} = 0 \ . \tag{1}$$

Here, $\mathbf{z}^{\pm}$ denote the Elsässer variables $\mathbf{z}^{\pm} = \mathbf{u} \pm \mathbf{B}$. (We assume a magnetic Prandtl number equal to one which places no restrictions on the analysis in the inertial range.) The quantities of interest are the p-th order structure functions

$$\left\langle |\Delta Z_\ell^{\pm}|^p \right\rangle = \left\langle |\mathbf{z}^{\pm}(\mathbf{x}+\mathbf{l}) - \mathbf{z}^{\pm}(\mathbf{x})|^p \right\rangle ,$$

which are expected to show scaling behavior in the inertial range $\eta \ll \ell \ll L$ spanned by the integral scale L (the scale at which energy is fed into the fluid) and the dissipation scale η at which the viscosity term in (1) becomes effective.

One important difference between hydro– and magnetohydrodynamic turbulence is the different interaction time τ_ℓ. Whereas in hydrodynamic turbulence the interaction time is solely governed by the nonlinearity, $\tau_{NL} \propto \ell/z_\ell$, in MHD the relation is modified due to the Alfvén time $\tau_A \propto \ell/B_0$ [1]. The energy transfer is slowed down by a factor of τ_{NL}/t_A because interactions occur only between small scale structures traveling in opposite directions (Alfvén waves) relative to the large scale magnetic field $\mathbf{B_0}$. Thus the interaction time is given by

$$\tau_\ell \propto \tau_{NL}^2/t_A \propto \ell B_0/z_\ell^2 \ . \tag{2}$$

Assuming that the energy dissipation is scale independent leads to the Kraichnan–Iroshnikov scaling [1, 2] for the structure functions of the Elsässer variables

$$\left\langle |\Delta Z_\ell^{\pm}|^p \right\rangle \propto \ell^{\zeta_p} \ , \ \ \zeta_p = \frac{p}{4} \ . \tag{3}$$

Satellite observations of MHD turbulence in the solar wind [3, 4, 5] show that the exponents ζ_p are nonlinear functions of p in contrast to the Kraichnan–Iroshnikov theory. The nonlinear behavior is astonishing well described by a phenomenological model [6] which is based on ideas introduced by She and Leveque [7]. Unfortunately, this nonlinear scaling could not yet be derived from the equations of motion directly even by the most sophisticated methods. Both, renormalized perturbation theory and renormalization group methods do not describe intermittency effects. The different promising approach of operator-product expansion initiated by Polyakov [8] is discussed in section 3. Even the interpretation of solar wind observations is still under question and therefore there is a need to perform numerical simulations in addition.

Direct numerical simulations are one possible approach to study energy spectra and structure functions. In [9] a projection method is used, where time-stepping is performed in a second order upwind manner in order to obtain an as large as possible inertial range. The numerically calculated structure functions show a similar behavior as in the phenomenological model [6] but the statistical certainty is very limited in this kind of simulations. As an alternative to direct numerical simulations in two-dimensional hydrodynamic flows, Breuer and Petruccione [10] recently developed a stochastic simulation method, that has proved to yield energy spectra showing a very clear power law behavior. In section 2 this stochastic approach is used to yield structure functions for magnetohydrodynamic flows.

2 Stochastic simulations

In the two-dimensional incompressible case the MHD equations can be formulated in scalar quantities namely the vorticity ω and the magnetic stream function ψ

$$\partial_t\, \omega + \mathbf{u} \cdot \nabla \omega \ = \ \mathbf{B} \cdot \nabla j + \nu \Delta \omega \, , \tag{4}$$
$$\partial_t\, \psi + \mathbf{u} \cdot \nabla \psi \ = \ \eta \Delta \psi \, . \tag{5}$$

The vorticity ω is given as the Laplacian of the kinetic stream function φ

$$\omega = \Delta \varphi \tag{6}$$

and the velocity is defined as $\mathbf{u} = \mathbf{e}_z \times \nabla \varphi$. Analogously, the magnetic field and current are given by $\mathbf{B} = \mathbf{e}_z \times \nabla \psi$ and $j = \Delta \psi$.

Here, we will first derive a discrete master equation and make it plausible by heuristic arguments. Then we proceed by showing that the equation of motion for the generating functional connected with this master equation fulfills the Hopf equation for MHD in the continuum limit. Finally, we discuss how to solve the master equation stochastically and show first results.

2.1 Master equation

The two-dimensional array $(0,1] \times (0,1]$ is discretized with $m \times m$ grid points. The vorticity and the kinetic and magnetic stream functions are defined on these grid points $(\omega, \varphi, \psi) := (\{\omega_{i,j}\}, \{\varphi_{i,j}\}, \{\psi_{i,j}\})$. The $(3\ m^2)$-dimensional phase space Γ is then defined as $\Gamma := \{(\omega, \varphi, \psi) |\ \omega_{i,j}, \varphi_{i,j}, \psi_{i,j} \in \mathbf{R}\}$. This allows for the representation of the system's state by a point in this space Γ. Assuming different realizations of the same stochastic process at each time a normalized probability distribution $P(\omega, \varphi, \psi)$ of the systems on Γ can be introduced.

The temporal evolution of this probability distribution is governed by a discrete master equation $\partial_t P(\omega, \varphi, \psi) = \mathcal{A} P(\omega, \varphi, \psi)$ with the time evolution operator

$$\mathcal{A} = \mathcal{A}_d^k + \mathcal{A}_d^m + \mathcal{A}_c^k + \mathcal{A}_c^m + \mathcal{A}_p\,. \tag{7}$$

The different terms making up the operator $\mathcal{A}$ will subsequently be described in detail.

First the damping term $\nu \Delta \omega$ in the equation for the vorticity (4) is modeled by the operator

$$\mathcal{A}_d^k = \frac{\nu}{\alpha_k \delta l^2} \sum_{i,j} \sum_{n=1}^{4} \left([\det(b_{i,j}^n)^{-1}] \mathcal{B}_{i,j}^n - I \right)$$

where the operator $\mathcal{B}_{i,j}^n$ acting on a function F on Γ is given as $\mathcal{B}_{i,j}^n F(\omega, \varphi, \psi) = F({b_{i,j}^n}^{-1}(\omega, \varphi, \psi))$. The parameter α_k measures the strength of the transition and δl denotes the spatial discretization. This notation that was chosen similar in Breuer and Petruccione's work on hydrodynamic flows [10] will be maintained in the following.

The map $b_{i,j}^1$ on Γ is defined as

$$b_{i,j}^1 : \begin{cases} \omega_{i,j} & \longmapsto \quad \omega_{i,j} \quad - \quad \alpha_k\, \omega_{i,j} \\ \omega_{i+1,j} & \longmapsto \quad \omega_{i+1,j} \quad + \quad \alpha_k\, \omega_{i,j} \end{cases}$$

and analogously with indices $n = 2, 3, 4$ for the other nearest neighbor cells $(i-1, j)$, $(i, j+1)$, $(i, j-1)$. The transfer of vorticity between neighboring points by application of $b_{i,j}^1$ is additionally illustrated in Figure 1. It now becomes obvious how the combined action of the four different mappings $(n = 1, 2, 3, 4)$ corresponds to the usual finite difference approximation of the Laplacian.

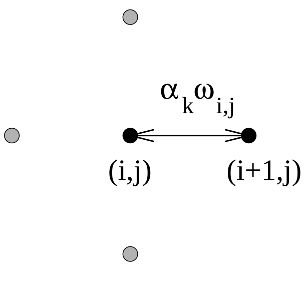

Figure 1: Action of $b_{i,j}^1$.

The damping term $\eta \Delta \psi$ in the equation for the magnetic stream function (5) can be treated in just the same way to give the operator $\mathcal{A}_d^m$.

Now the nonlinear terms in the vorticity equation (4) will be discussed. The convective term $\mathbf{u} \cdot \nabla \omega$ can explicitly be written as $\partial_x(\omega \partial_y \varphi) - \partial_y(\omega \partial_x \varphi)$. Here we will focus on its first part $\partial_x(\omega \partial_y \varphi)$ that is modeled by the operator $\mathcal{A}_c^k$ in the master equation

$$\mathcal{A}_c^k = \frac{1}{2\gamma \delta l} \sum_{i,j} \sum_{n} \left([\det(c_{i,j}^n)^{-1}] \mathcal{C}_{i,j}^n - I \right)$$

with

$$c_{i,j}^1 : \begin{cases} \omega_{i,j} & \longmapsto \quad \omega_{i,j} \quad - \quad \gamma_k\, N_{i,j}^1 \\ \omega_{i+1,j} & \longmapsto \quad \omega_{i+1,j} \quad + \quad \gamma_k\, N_{i,j}^1 \end{cases}$$

and $N_{i,j}^1 = \omega_{i,j}\ d_y \varphi_{i,j} + \omega_{i+1,j}\ d_y \varphi_{i+1,j}$ where $d_y \varphi_{i,j} = (\varphi_{i,j+1} - \varphi_{i,j-1})/(2\delta l)$. The action of the map $c_{i,j}^1$ is shown in Figure 2. Again the involved grid points are shown in black.

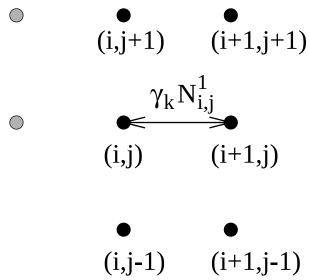

Figure 2: Action of $c_{i,j}^1$.

The second part of the convective term $-\partial_y(\omega \partial_x \varphi)$ is analogously treated by a map $c_{i,j}^2$ where x- and y-derivatives are interchanged and the sign is reversed.

The other nonlinear terms $-\partial_x(\Delta\psi\ \partial_y \psi)$ and $\partial_y(\Delta\psi\ \partial_x \psi)$ in the equation of vorticity (4) are treated in the same way. The convective term in the equation for the magnetic stream function (5) is handled just analogously to give $\mathcal{A}_c^m$.

The relation $\Delta\varphi = \omega$ is modeled by the constraint $\partial_t \varphi = (\Delta\varphi - \omega)/\epsilon$, where the error is determined by the value of the parameter ϵ. Obviously, this can be done by

$$\mathcal{A}_p = \frac{1}{\beta \epsilon} \sum_{i,j} \left([\det(d_{i,j})^{-1}] \mathcal{D}_{i,j} - I \right)$$

with $d_{i,j} : \varphi_{i,j} \ \longmapsto\ \varphi_{i,j}\ +\ \beta\,(D\varphi_{i,j} - \omega_{i,j})$ where

$$D\varphi_{i,j} = \frac{\varphi_{i+1,j} + \varphi_{i,j+1} + \varphi_{i-1,j} + \varphi_{i,j-1} - 4\varphi_{i,j}}{\delta l^2}\,.$$

Figure 3 shows the grid points involved in this mapping.

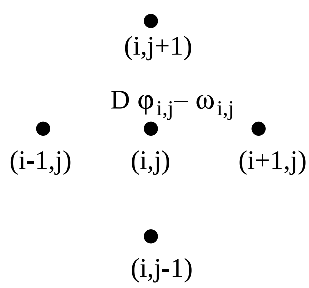

Figure 3: Action of $d_{i,j}$.

Now, having defined all the necessary operators we want to show that a solution of this stochastic process in the continuum limit fulfills the MHD-analogue of the Hopf equation. (Introducing and temporally differentiating the generating functional one derives the Hopf equation from equations (4),(5) and (6), see e.g. McComb [11].)

The temporal evolution of the expectation value of an arbitrary function $F = F(\omega, \varphi, \psi)$ is

$$\partial_t \langle F \rangle = \langle [F, \mathcal{A}] \rangle$$

where [,] denotes the commutator and where the expectation value is defined as

$$\langle F \rangle = \int \prod_{i,j} d\omega_{i,j} d\varphi_{i,j} d\psi_{i,j} \; F(\omega, \varphi, \psi) P(\omega, \varphi, \psi, t) \; .$$

The discretized generating functional is given as

$$\Phi(z_\omega, z_\varphi, z_\psi, t) = \left\langle e^{i\delta l^2 \sum_{i,j} (z_{\omega\, i,j} \omega_{i,j} + z_{\varphi\, i,j} \varphi_{i,j} + z_{\psi\, i,j} \psi_{i,j})} \right\rangle \quad .$$

In the limit of continuous space $\delta l \longrightarrow 0$ and expanding to $\mathcal{O}(\alpha_{k,m} \delta l^2)$ for the temporal evolution of the generating functional one gets

$$\begin{aligned}
\frac{\partial \Phi[z_\omega, z_\varphi, z_\psi, t]}{\partial t} &= i \int dxdy \; [z_\omega - \Delta^{-1} z_\varphi] \\
&\times \left(\nu \Delta \frac{\delta}{i \delta z_\omega} + \left[\frac{\partial}{\partial x} \frac{\delta}{i\delta z_\omega} \frac{\partial}{\partial y} \frac{\delta}{i \delta z_\varphi} \; - \; \frac{\partial}{\partial y} \frac{\delta}{i\delta z_\omega} \frac{\partial}{\partial x} \frac{\delta}{i \delta z_\varphi} \right] \right. \\
&+ \left. \left[- \frac{\partial}{\partial x} \Delta \frac{\delta}{i \delta z_\psi} \frac{\partial}{\partial y} \Delta \frac{\delta}{i \delta z_\varphi} \; + \; \frac{\partial}{\partial y} \Delta \frac{\delta}{i \delta z_\psi} \frac{\partial}{\partial x} \Delta \frac{\delta}{i \delta z_\varphi} \right] \right) \\
&\Phi[z_\omega, z_\varphi, z_\psi, t] \\
&- \nu \, \alpha_k \, \delta l^2 \int dxdy \, |\nabla z_\omega|^2 \, \frac{\delta^2}{i^2 \delta z_\omega^2} \Phi[z_\omega, z_\varphi, z_\psi, t] \\
&+ i \int dxdy \; z_\psi \\
&\times \left(\eta \Delta \frac{\delta}{i \delta z_\psi} + \left[\frac{\partial}{\partial x} \frac{\delta}{i\delta z_\psi} \frac{\partial}{\partial y} \frac{\delta}{i \delta z_\varphi} \; - \; \frac{\partial}{\partial y} \frac{\delta}{i\delta z_\psi} \frac{\partial}{\partial x} \frac{\delta}{i \delta z_\varphi} \right] \right) \\
&\Phi[z_\omega, z_\varphi, z_\psi, t] \\
&- \eta \, \alpha_m \, \delta l^2 \int dxdy \, |\nabla z_\psi|^2 \, \frac{\delta^2}{i^2 \delta z_\psi^2} \Phi[z_\omega, z_\varphi, z_\psi, t] \quad .
\end{aligned}$$

This equation is identical to the Hopf equation for the MHD case except for two additional terms namely the second and fourth integral. They vanish in the limit of continuous space and small parameters α_k and α_m. That means in this limiting case the master equation (7) yields a complete description of the stochastic properties of the turbulent fields. The additional terms originate from fluctuations introduced by the probabilistic time evolution. Their effect on the energy spectra can easily be removed by an appropriate renormalization procedure as already shown by Breuer and Petruccione [10] in the hydrodynamic case.

2.2 Numerical simulations

In this section we will briefly explain how numerical realizations of the described stochastic process are performed. Of course, first of all, a sensible initial condition for φ and ψ has to be chosen. Then until reaching the final time the following procedure is repeated. First, there are two random choices at each time. One random number is needed to find the location (i, j) where the next transition will take place and another one to find out which kind of transition is selected. The maps discussed above, $b^n_{i,j}$, etc. are weighted with the probability corresponding to their transition rates. Afterwards the transition is carried out. The time scale of these operations is characterized by the inverse transition rates and is therefore strongly dependent on the choice of the parameters α_k, α_m, γ_k, γ_m, β and ϵ. This consideration already reveals the crucial fact that when steep gradients are forming, very small parameters for the transfers are needed to obtain numerical stability. Unfortunately, this restriction leads to high computational costs. Considering the following numerical results one should keep in mind that for this reason the Reynolds numbers of the actual numerical realizations are very moderate.

In Figure 4 we show an example of the indirect evaluation of structure functions by means of extended self-similarity (ESS) as introduced by Benzi et al [12]. Finally, in Figure 5 we

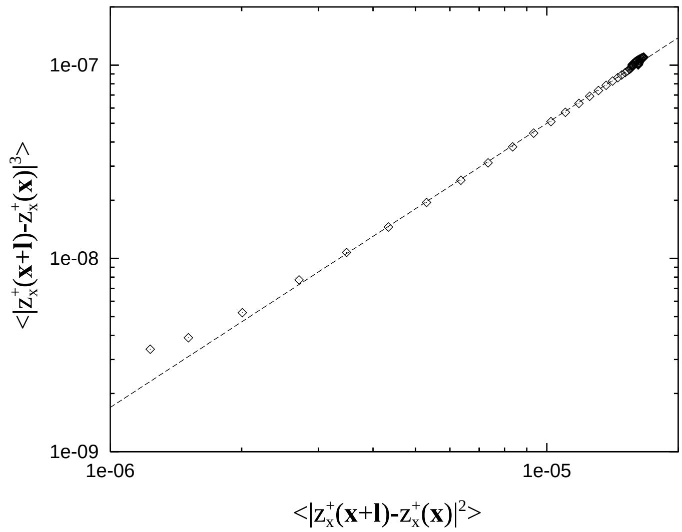

Figure 4: Third order structure function versus second order.

give the resulting scaling exponents ζ_p on the assumption that $\zeta_4 = 1$. Keeping in mind all the uncertainties, nevertheless

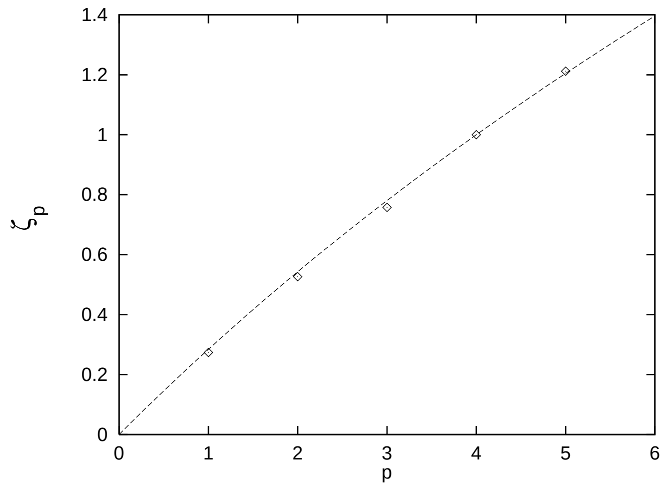

Figure 5: Scaling exponents ζ_p versus order p.

these results may be regarded as a further indication for the validity of such kind of scaling behavior as predicted by the phenomenological theory [6] and shown as a dashed line in this figure.

3 Operator-product expansion

Until now, analytic attempts to calculate structure functions for turbulent flows (hydro- and magnetohydrodynamic flows) failed to describe intermittency corrections. Therefore it was a pleasant welcome when in 1993 Polyakov [8] tried to apply conformal field theory to the problem of 2D Navier–Stokes turbulence. The validness of a conformal field theory for the inverse cascade was very soon questioned by Falkovich and Hanany [13]. Especially, it was shown that minimal models are not compatible with the infinite number of vorticity integrals present in the 2D Navier–Stokes equations. As we will show, not only the inverse cascade but also a possible direct cascade can not be described by a finite number of primary fields.

Still we believe that one ingredient of conformal field theory, the operator-product expansion (OPE), is a fruitful concept for turbulence theories. If one assumes the validness of an OPE, then a trivial consequence of the energy flux to small scales is the well known exact relation $\zeta_3 = 1$ for the third order structure function in Galilean invariant systems. Another more interesting observation is that a reinterpretation of Kolmogorov's refined similarity hypotheses follows from the OPE. In addition, Burgers turbulence can be handled with OPE due to the presence of infinitely many conserved quantities.

In the inertial range, correlation functions are assumed to be homogeneous functions. Let ψ_i denote some scaling function (derived from the velocity field) then this means

$$\langle\psi_1(x_1)\cdots\psi_n(x_n)\rangle = \ell^{-\Delta_{\psi_1}}\cdots\ell^{-\Delta_{\psi_n}}\,\langle\psi_1(\frac{x_1}{\ell})\cdots\psi_n(\frac{x_n}{\ell})\rangle \ .$$

If we want to calculate correlation functions which contain composite fields, e.g. $\psi^2(x)$, then the correlation function would diverge. The information of this singular behavior is contained in the operator-product expansion of Wilson [14] which reads

$$\psi_k(x+a)\psi_l(x) = \sum f_{kl}^m |a|^{\Delta_{\psi_m}-\Delta_{\psi_k}-\Delta_{\psi_l}}\psi_m(x) + \text{secondaries} \ .$$

(For an extensive treatment of the operator-product expansion see Collins [15].)

In order to fulfill the inviscid Hopf equation, Polyakov [8] introduced the *point splitted* version of the nonlinear convection term

$$u_\alpha(x)\partial_\alpha u_\beta(x) := \overline{\lim_{a\to 0}}\, u_\alpha(x+a)\partial_\alpha u_\beta(x) \ .$$

He also concluded that in the 2D Navier–Stokes equations a steady turbulent state with a constant enstrophy flux to small scales should result in the relation

$$\langle\dot\omega(x+\ell)\omega(x)\rangle \propto \ell^0$$

which is equivalent to

$$\Delta_u + \Delta_{u^2} = -3 \ . \tag{8}$$

Here, ω denotes the vorticity and Δ_{u^2} the scaling dimension of the renormalized composite field u^2. Recently, Benzi et al [16] tested this relation (8) by direct numerical simulation. Although they want to confirm conformal field theory, they actually try to confirm the validness of the OPE which is independent of conformal symmetry.

In a steady turbulent state with constant energy flux to small scales one obtains

$$\langle\dot{u_\alpha}(x+\ell)u_\alpha(x)\rangle \propto \ell^0$$

and therefore the relation

$$\Delta_u + \Delta_{u^2} = -1 \ . \tag{9}$$

This relation (9) has been tested in 2D magnetohydrodynamic turbulence. The numerical procedure is described in [9] but to obtain better statistics, an appropriate external driving has been added. From the definition of Δ_u follows

$$\langle u_\alpha(x+\ell)u_\alpha(x)\rangle \propto \ell^{-2\Delta_u}$$

so that the energy spectrum E_k reads

$$E_k \propto k^{2\Delta_u - 1} =: k^{-\gamma} \ .$$

Using the OPE one arrives at

$$\langle\dot\omega(x+\ell)\dot\omega(x)\rangle \propto \ell^{-2\Delta_{u^2}-4} = \ell^{2\Delta_u - 2}$$

so that the spectrum Ω_k of this quantity scales like $k^{-2\Delta_u+1} = k^\gamma$, which has the opposite slope of the energy spectrum.

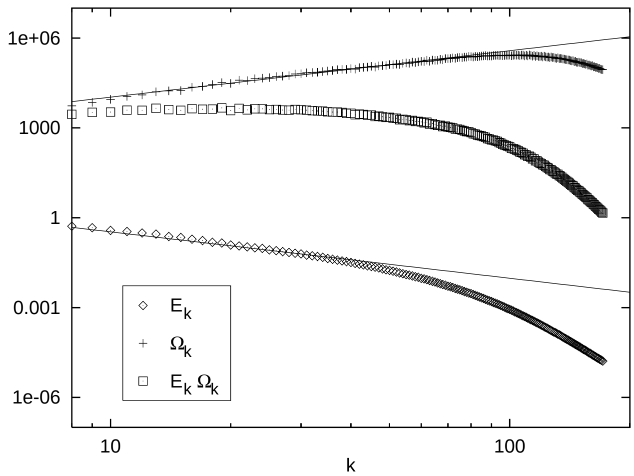

Figure 6: Energy spectrum E_k and spectrum Ω_k of $\dot\omega^2$. The middle graph depicts the product $E_k\ \Omega_k$. The slopes of the asymptotic lines correspond to the value of the phenomenological model in [6].

This behavior is depicted in Figure 6. Although the inertial range is still rather limited using 512×512 mesh points, these results do support the correctness of an OPE.

Next, we want to clarify the connection between correlation functions and higher order structure functions

$$\langle|u(x+\ell)-u(x)|^p\rangle \propto \ell^{\zeta_p} \ .$$

Let us therefore calculate the first few ones. In the second order structure function, the only relevant term is proportional to $u(x+\ell)u(x)$ with scaling $\ell^{-2\Delta_u}$. In the third order structure function, scaling is determined by $u(x+\ell)u(x)^2$ with scaling $\ell^{-\Delta_u-\Delta_{u^2}}$. Here we see that for a direct cascade characterized by eqn.(9) the OPE implies

$$\zeta_3 = 1 \ , \tag{10}$$

which is known as an exact relation [17]. For the fourth order structure function two terms remain: $u(x+\ell)u(x)^3$ and $u(x+\ell)^2u(x)^2$. Assuming a direct cascade, so that the relevant length scale for applying Schwarz inequality is the dissipation length (and not the integral scale) [18], implies that the dominating term is the second one: $u(x+\ell)^2u(x)^2$. In general it holds that the p-th order structure function is related to the OPE via

$$\zeta_{2p} = -2\Delta_{u^p} \quad , \quad \zeta_{2p+1} = -(\Delta_{u^{p+1}} + \Delta_{u^p}) \quad . \tag{11}$$

This relation shows that if turbulence could be described by some minimal model (which means only a finite number of scaling dimensions) then ζ_p would be constant for p sufficiently large. This is a situation not compatible with experiments in hydro- and magnetohydrodynamic flows.

3.1 Kolmogorov's refined similarity hypothesis

Using the above observation (11) one can derive a relation between the structure functions of the energy transfer and the structure functions of the velocity increments. Therefore, we define the energy transfer as $\Pi(x) = (u(x) \cdot \nabla u(x)) \cdot u(x)$ and assume that the structure functions of the energy transfer scale as

$$\langle |\Pi(x+\ell) - \Pi(x)|^p \rangle \propto \ell^{\tau_p} \tag{12}$$

in the inertial range. For both, even and odd structure functions follows

$$\begin{aligned} \tau_{2p} &= -2\Delta_{u^{3p}} - 2p \\ &= -2p + \zeta_{6p} \\ \tau_{2p+1} &= -\Delta_{u^{3p+2}} - \Delta_{u^{3p+1}} - (2p+1) \\ &= -(2p+1) + \zeta_{6p+3} \quad , \end{aligned}$$

so that we obtain

$$\zeta_{3p} = p + \tau_p \quad . \tag{13}$$

This is Kolmogorov's refined similarity hypothesis where the energy dissipation averaged over a ball of size ℓ has been substituted by the differences of the energy transfer across a distance ℓ.

3.2 Structure functions in Burgers turbulence

As the energy conservation already led to a relation between the scaling dimensions Δ_u and Δ_{u^2}, additional integrals of motion may provide further relations which can finally close the algebra as in the case of Burgers turbulence. Burgers equation

$$u_t + uu_x = 0$$

has infinitely many integrals of motion $\frac{d}{dt}\int u^{n+1}dx = 0$. Assuming that these integrals cascade to small scales, supplies the relations

$$\Delta_{u^2} + 1 + \Delta_{u^n} = 0 \tag{14}$$

originating from the nonlinearity. From this follows that $\Delta_{u^n} = -1/2$ which means that the exponents of the structure functions fulfill

$$\zeta^p = 1 \quad \text{for} \quad p \geq 1 \tag{15}$$

which is known from the bifractal interpretation of Burgers turbulence [19].

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