Current Sheet Formation in 3D Ideal Incompressible Magnetohydrodynamics

Rainer Grauer[1] and Christiane Marliani[2]
[1] *Institut für Theoretische Physik I, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany*
[2] *Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012-1185, USA*

The evolution of current density and vorticity in the ideal, inviscid incompressible magnetohydrodynamic equations in three dimensions is studied numerically. Highly effective resolutions are obtained by adaptive structured mesh refinement techniques. We report on results for three different initial conditions showing similar behavior: in the early stage of the evolution a fast increase in vorticity and current density is observed. Thereafter, the evolution towards nearly two-dimensional current sheets results in a depletion of nonlinearity.

PACS number(s): 52.30.-q, 52.65.Kj, 47.11.+j, 03.40.Gc

Recently, remarkable progress has been made in calculating the tails of the probability distribution (PDF) of velocity increments in the model system of Burgers turbulence. The very far tails (where dissipation is important) could be described by the instanton approach [1]. The main contribution to the PDF originates from shocks smoothed by dissipation. For the medium far tails, a Lagrangian description was successful [2,3]. There, the preshocks represent the main contribution to the PDF. Both attempts highlight the importance of singular structures which the system tries to develop. Singular structures were already an important input in the Frisch-Parisi-approach [4] of intermittency. The behavior of high-order structure functions is governed by the spatial scaling (singularity strength) of the singular structures and their codimension. She and Leveque [5] and Dubrulle [6] added some further assumptions on the temporal scaling and how to interpolate structure functions of different order (which is equivalent to the assumption of a log-Poisson distribution). They obtained a model where all parameters were physically accessible and where the prediction of the model is indistinguishable from various measurements of turbulent flows. This model could also be applied to magnetohydrodynamic flows [7,8]. There, the singular structures are not vortex tubes as in the Navier–Stokes-equation but current sheets. Still open is the question how Alfvén waves alter the temporal scaling. Recently, Politano and Pouquet [9] found an analogue to the 4/5 law for the MHD-equations suggesting that Alfvén waves play a less important role than previously assumed [10,11]. However, numerical simulations of MHD turbulence show a non-coherent picture. Two-dimensional simulations [12] show a much more intermittent behavior than corresponding three-dimensional simulations [13]. The latter could well be described by assuming current sheets as the most singular structures and neglecting completely the effect of Alfvén waves. The question why two- and three-dimensional simulations differ so much is still an open problem. Nevertheless, the information that the most singular structures are current sheets is a main ingredient for understanding structure functions in the limit of very high order.

An additional important question is whether these structures, i.e. vortex tubes or current sheets, will become singular in finite time. For the Burgers shocks this is easy to answer, however for the other problems this is still an open mathematical problem. In addition, only few mathematical theorems are available to guide the numerics. The most important one is the Beale-Kato-Majda result [14,15] which states that for a global solution the time integral over the L_∞-norm must stay finite. This implies that if there is a power law blow up, then the minimum exponent is one. For the ideal incompressible Euler equations there is growing numerical evidence [16–19] that a finite time singularity in the vorticity develops. For the three-dimensional MHD equations the situation is even more subtle and less well studied.

In all those studies, the main numerical difficulty is the insufficient resolution. To overcome this problem we use the technique of adaptive structured mesh refinement (AMR). Our algorithm for this purpose is very similar to the one by Berger and Colella [20] and was recently described in Friedel *et al.* [21] and applied successfully to the case of the incompressible Euler equations [19]. With this technique it is possible to achieve effective resolutions of up to 4096^3 mesh points on medium sized supercomputers.

In this letter, we consider three-dimensional magnetohydrodynamic flows. The time evolution of velocity and magnetic field is governed by the following equations

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \mathbf{B}\cdot\nabla\mathbf{B} - \nabla p \ ,$$
$$\mathbf{B}_t + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u}$$

where p denotes the sum of kinetic and magnetic pressure and velocity $\mathbf{u}$ and magnetic field $\mathbf{B}$ are divergence-free. The symmetry of the equations becomes apparent by transformation to Elsässer variables $\mathbf{z}^\pm = \mathbf{u} \pm \mathbf{B}$

$$\partial_t \mathbf{z}^\pm + \mathbf{z}^\mp\cdot\nabla\mathbf{z}^\pm + \nabla p = 0 \ , \ \ \mathrm{div}\,\mathbf{z}^\pm = 0 \ . \qquad (1)$$

This setting of the equation will be useful later on in order to understand the depletion of nonlinearity. In all simulations, periodic boundary conditions are used.

The study of vortex and current dynamics for the three-dimensional MHD equations was triggered by Politano *et al.* [22]. They considered a three-dimensional generalization of the so-called Orszag-Tang initial condition which has similar symmetries as the Taylor-Green vortex for the Euler equations. The initial velocity $\boldsymbol{u}_0$ and magnetic field $\boldsymbol{B}_0$ read

$$\mathbf{u}_0 = \begin{pmatrix} -2\sin x_2 \\ 2\sin x_1 \\ 0 \end{pmatrix}, \mathbf{B}_0 = 0.8 \begin{pmatrix} -2\sin(2x_2) + \sin x_3 \\ 2\sin x_1 + \sin x_3 \\ \sin x_1 + \sin x_2 \end{pmatrix}. \tag{2}$$

The absolute value of the current density is visualized in Fig. 1.

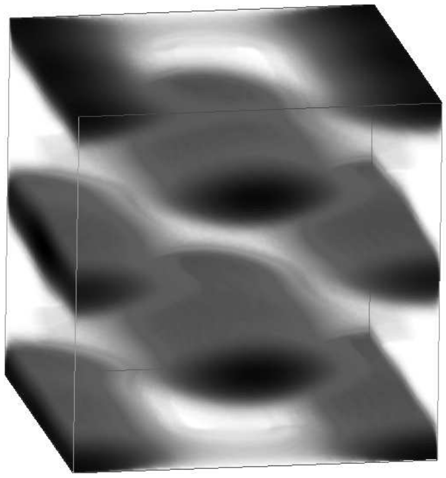

FIG. 1. Volume rendering of $|\boldsymbol{j}|$ at time 0..

Their findings were that current sheets develop and that the growth of current density and vorticity is at most exponential in time. Their simulation reached a resolution of 180^3 collocation points with a pseudo-spectral scheme.

We redid their simulation using AMR and could completely confirm the nonadaptive treatment in that respect that quasi two-dimensional current sheets are formed associated with only exponential growth in current density and vorticity. Fig. 2 shows a volume rendering of the absolute value of current density at time 0.6 when 4 levels of refinement each with a factor of 2 are present. To provide the same local resolution a uniform mesh with 1024^3 grid points would be needed.

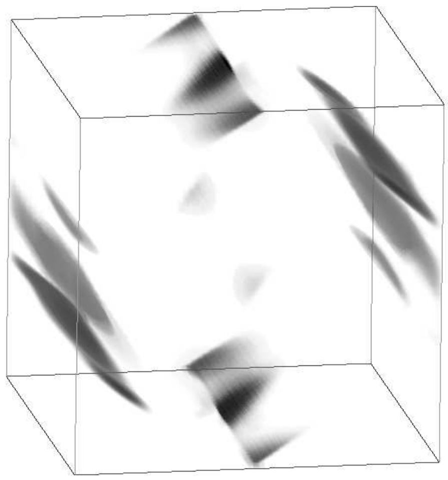

FIG. 2. Volume rendering of $|\boldsymbol{j}|$ at time 0.6.

The sum of magnetic and kinetic energy is conserved within less than 0.5%. Fig. 3 shows the temporal evolution of the L^∞-norms of current density and vorticity. As seen in the plot current density and vorticity settle into an exponential regime resembling the dynamics of MHD in two dimensions, compare [21].

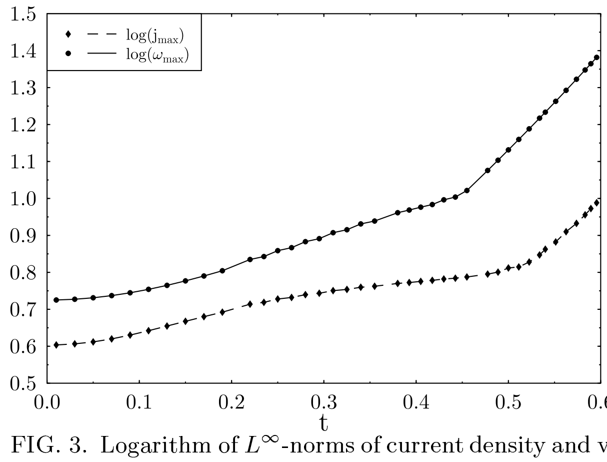

FIG. 3. Logarithm of L^∞-norms of current density and vorticity over time .

In order to study a less symmetric initial condition, we looked for the temporal evolution of slightly perturbed parallel flux tubes. In the initial stage, the flux tubes attract each other as expected (see Figure 4). A fast increase of current density is observed during this initial stage. Due to the compression of the two flux tubes a current sheet is formed and again the dynamics tends to be nearly two-dimensional. In this stage, the temporal growth of current density reduces to exponential which is completely compatible with the tendency of the plasma to become almost two-dimensional. The final current sheet is shown in Figure 5. This pancake like structure is a magnification of the central part of Figure 4. An effective local resolution comparable to 4096^3 mesh points on a uniform grid was reached.

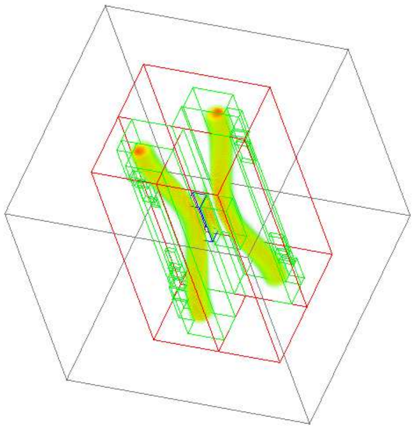

FIG. 4. Volume rendering of $|\boldsymbol{j}|$ at time 0.42.

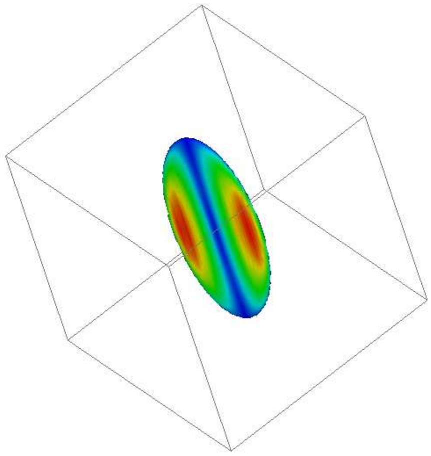

FIG. 5. Volume rendering of $|\boldsymbol{j}|$ at time 0.57.

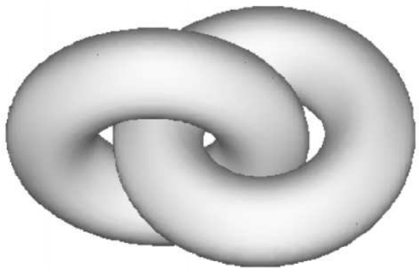

FIG. 6. Volume rendering of $|\boldsymbol{j}|$ at time 0.0.

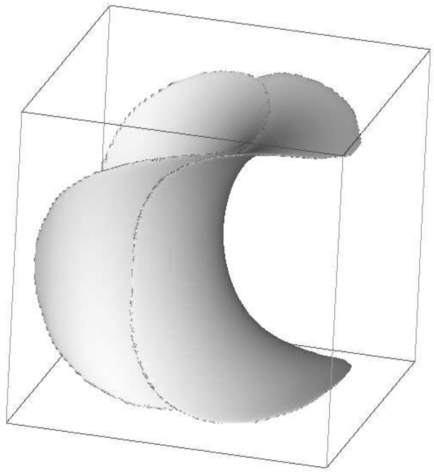

FIG. 7. Volume rendering of $|\boldsymbol{j}|$ at time 1.04.

Finally, we discuss results for a third type of initial conditions consisting of two linked flux rings. A very similar configuration has recently been studied by Kerr and Brandenburg [23]. Their simulations were performed with a pseudo-spectral method using 384^3 modes. The authors claim to observe a finite time singularity in the current density. We could not confirm their results. We find a dramatic increase of current density in the early stages until almost two-dimensional current sheets have formed (see Figure 7). Although these current sheets are not flat, but have the form of potato chips (with finite macroscopic curvature), the growth reduces again to exponential. This is demonstrated in Figure 8, where the growth of current density is compatible with an exponential fit. The behavior differs from two-dimensional MHD in that the growth of current density is dramatic in the early stage whereas in 2D MHD the growth is exponential nearly from the beginning.

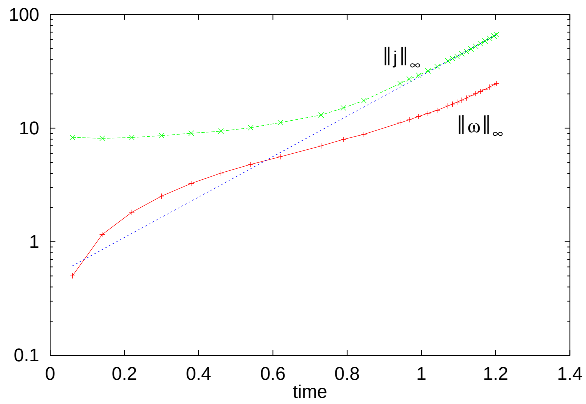

FIG. 8. Growth of vorticity and current density.

It is also interesting to note that it is impossible to fit current density and vorticity with a power law blow-up $\approx c/(T-t)$, where the singularity time is equal for both, current density and vorticity. In our opinion, this makes it even less plausible that a finite time singularity develops.

The exponential growth of the current density can be partially explained by the depletion of nonlinearity through the geometry of the current sheet. Taking the curl of the ideal MHD equations (1) yields

$$\partial_t \boldsymbol{\omega}^{\pm} + \mathbf{z}^{\mp} \cdot \nabla \boldsymbol{\omega}^{\pm} = \boldsymbol{\omega}^{\pm} \cdot \nabla \mathbf{z}^{\mp} + \sum_i \nabla z_i^{\pm} \times \nabla z_i^{\mp} \quad (3)$$

with $\boldsymbol{\omega}^{\pm} = \nabla \times \mathbf{z}^{\pm}$ meaning the sum and difference of vorticity $\boldsymbol{\omega}$ and current density $\mathbf{j}$ and with $z_i^{\pm}$ the i-th component of the vector $\mathbf{z}^{\pm}$ (see Pouquet [24]). Now it becomes clear that if two-dimensional current sheets develop the derivatives in two directions remain smooth and only the derivative transverse to the sheet contributes to the right most term in the vorticity and current density production. This means that at least this production term acts like a linear term. Unclear is the role of the first production term $\boldsymbol{\omega}^{\pm} \cdot \nabla \mathbf{z}^{\mp}$ on the right hand side of equation (3). For the case of slightly perturbed parallel flux tubes, $\boldsymbol{\omega}$ and $\mathbf{j}$ both point in the z-direction and are orthogonal to the direction of the steepest gradient. The situation is not as clear for the case of the linked flux tubes. However, this term is similar to the production term for vorticity in the Euler equations and we point out that no one has seen a finite time singularity in the Euler equations for sheet like structures. We think and hope that this issue can analytically be proven as in the case of quasi-geostrophic flows, where Cordoba [25] could rule out a finite time blow up under minimal assumptions. Since this technique could also be translated to the two-dimensional MHD case [26] and since the geometry of the presented current sheets is rather simple an application of this technique seems to be very promising.

ACKNOWLEDGMENTS

We like to acknowledge interesting discussions with M.J. Berger, K. Germaschewski, K.H. Spatschek and H. Strauss. This work was performed under the auspices of the Sonderforschungsbereich 191. C. Marliani was supported by U.S. DOE Grants No. DE-FG02-88ER25053 and DE-FG02-86ER53223. Computations were performed on SGI CRAY Origin2000 at the Computing Center of the Heinrich-Heine-University and at the National Center for Supercomputing Applications, University of Illinois at Urbana-Champaign under Grant No. DMS980009N.

[1] E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett. **78**, 1452 (1997).
[2] W. E, K. Khanin, A. Mazel, and Y. Sinai, Phys. Rev. Lett. **78**, 1904 (1997).
[3] W. E. and E. V. Eijnden, submitted to Phys. Rev. Lett. (1999), chao-dyn/9901006.
[4] U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, edited by M. Ghil, R. Benzi, and G. Parisi (North–Holland, New York, 1985), pp. 84–88.
[5] Z.-S. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994).
[6] B. Dubrulle, Phys. Rev. Lett. **73**, 959 (1994).
[7] R. Grauer, J. Krug, and C. Marliani, Physics Lett. A **195**, 335 (1994).
[8] H. Politano and A. Pouquet, Phys. Rev. E **52**, 636 (1995).
[9] H. Politano and A. Pouquet, Phys. Rev. E **57**, R21 (1998).
[10] P. S. Iroshnikov, Sov. Astron. **7**, 566 (1964).
[11] R. H. Kraichnan, Phys. Fluids **8**, 1385 (1965).
[12] H. Politano, A. Pouquet, and V. Carbone, Europhysics Letters **43**, 516 (1998).
[13] W.-C. Müeller and D. Biskamp (unpublished).
[14] J. T. Beale, T. Kato, and A. Majda, Comm. Math. Phys. **94**, 61 (1984).
[15] G. Ponce, Comm. Math. Phys. **98**, 349 (1985).
[16] R. M. Kerr, Phys. Fluids A **5**, 1725 (1993).
[17] O. N. Boratav and R. B. Pelz, Phys. Fluids **6**, 2757 (1994).
[18] R. B. Pelz and Y. Gulak, Phys. Rev. Lett. **79**, 4998 (1997).
[19] R. Grauer, C. Marliani, and K. Germaschewski, Phys. Rev. Lett. **80**, 4177 (1998).
[20] M. J. Berger and P. Colella, J. Comput. Phys. **82**, 64 (1989).
[21] H. Friedel, R. Grauer, and C. Marliani, J. Comput. Phys. **134**, 190 (1997).
[22] H. Politano, A. Pouquet, and P. L. Sulem, Phys. Plasmas **2**, 2931 (1995).
[23] R. M. Kerr and A. Brandenburg, Phys. Rev. Lett. **83**, 1155 (1999).
[24] A. Pouquet, in *Plasma Astrophysics*, No. 468 in *Lecture Notes in Physics*, edited by C. Chiuderi and G. Einaudi (Springer-Verlag, Berlin, Heidelberg, 1996), pp. 163–212.
[25] D. Cordoba, Ann. of Math. **148**, 1135 (1998).
[26] D. Cordoba and C. Marliani, to appear in CPAM.