An energy estimate for a perturbed Hasegawa–Mima equation

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Abstract. It is commonly believed that drift waves and drift-wave turbulence play a major role in the understanding of anomalous transport at the plasma edge of a tokamak fusion reactor. A one-field equation describing the electrostatic potential fluctuations in this regime is the so-called Hasegawa–Mima equation. If this equation is driven by some instability and damped by some hyperviscous term, the energy grows exponentially in time which is not consistent with the approximations made in the derivation of the equation. Numerical simulations of a perturbed Hasegawa–Mima equation which includes in addition a so-called $E \times B$ nonlinearity showed that the energy saturates at a finite level. In this paper this numerical observation is proven analytically.

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1. Introduction

An open and still controversial issue in tokamak fusion reactors is the problem of anomalous transport associated with confinement degradation. One commonly believed explanation is the turbulent dynamics of electrostatic drift waves. The simplest nonlinear model describing the time evolution of drift waves was derived by Hasegawa and Mima [1]

$$\partial_t(1-\Delta)\phi = -\kappa\partial_y\phi + \{\phi, \Delta\phi\} + \mu\Delta^3\phi \quad . \tag{1}$$

Here ϕ describes the electrostatic potential fluctuations and density fluctuations n assuming a Boltzmann relation $n \sim \phi$. This equation also arises in the context of Rossby waves in the atmospheres of rotating planets [2].

If this equation (1) is supplemented by some driving $-\alpha\partial_{yy}\phi$ and damping $-(-\Delta)^k\phi$ it was shown in numerical experiments by Kono and Miyashita [3] that a large dipole evolves due to the existence of an inverse energy cascade. Unfortunately, this dipole grows nearly exponentially in time without saturation indicating that some relevant physical process is missing. This problem can be solved if one allows for a phase shift between density and potential fluctuations leading to a so-called $E \times B$ nonlinearity (see Liang *et al.* [4], Naulin *et al.* [5] and references therein). Numerical simulations of a model including this additional nonlinearity now show a saturation of energy at a finite level (see, e.g. Naulin *et al.* [5], Figure 3).

In this paper, the two-dimensional equation derived by Liang *et al.* [4] is studied as a prototype. This equation reads

$$\partial_t(1-\Delta)\phi = -\kappa\partial_y\phi - \alpha\partial_{yy}\phi + \{\phi, \Delta\phi\} + \beta\{\phi, \partial_y\phi\} + \mu\Delta^3\phi \quad , \tag{2}$$

with $\alpha > 0$, $\beta > 0$, $\mu > 0$ and where $\{\cdot\,,\cdot\}$ denotes the Poisson bracket

$$\{f,g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g).$$

The terms $-\kappa\partial_y\phi$, $-\alpha\partial_{yy}\phi$ and $\mu\Delta^3\phi$ describe a drift, a driving through an instability and damping, respectively. The nonlinear terms comprise a convection term $\{\phi,\Delta\phi\}$ and the additional $E\times B$ nonlinearity $\beta\{\phi,\partial_y\phi\}$. For technical reasons, hyperviscous damping $\mu\Delta^3\phi$ is used instead of viscous dissipation $-\mu\Delta^2\phi$. This is no severe restriction since energy flows to large scales.

The field ϕ is subjected to periodic boundary conditions in y

$$\phi(x,y) = \phi(x,y+L_y) \tag{3}$$

and to Dirichlet boundary conditions in x

$$\phi(0,y) = \phi(L_x,y) = 0 \ . \tag{4}$$

This corresponds to functions periodic with period $2L_x$ and antisymmetric in x, periodic in y and defined on $\Omega = [-L_x, L_x]\times[0,L_y]$.

Equation (2) implies that the spatial average $\int\phi d\Omega$ is constant in time, hence it can be set to zero without restriction. The identically vanishing solution is unstable for parameters α and μ and wavenumbers k_x and k_y satisfying

$$\alpha k_y^2 > \mu\left[k_x^2 + k_y^2\right]^3 \ . \tag{5}$$

The interesting regime considered here corresponds to the highly unstable case. As one can easily see, periodic boundary conditions in both directions do not make much sense since the nonlinearities vanish for ϕ chosen to be

$$\phi = \cos(k(x+y)) \ .$$

If $k_x = k_y = k$ satisfies the inequality (5), then ϕ grows exponentially in time.

The problem in deriving an estimate for ϕ can be seen by multiplying equation (2) with ϕ and integrating over space. The result is given by

$$\frac{1}{2}\partial_t\int\left(\phi^2 + (\nabla\phi)^2\right)d\Omega = \alpha\int(\partial_y\phi)^2 d\Omega - \mu\int(\nabla\Delta\phi)^2\,d\Omega \ . \tag{6}$$

The best one can obtain from equation (6) is an exponential growth in time as would be the case for the linearly unstable system.

The situation is very similar to the case of the one-dimensional Kuramoto–Sivashinsky equation

$$\partial_t u = -\partial_x^2 u + u\partial_x u - \partial_x^4 u \ . \tag{7}$$

Multiplying equation (7) with u and integrating over space yields

$$\frac{1}{2}\partial_t\int u^2 dx = \int(\partial_x u)^2 dx - \int(\partial_x^2 u)^2 dx \ , \tag{8}$$

which corresponds to the simple estimate (6). Physically, the stabilizing effect in the Kuramoto–Sivashinsky equation can be readily understood by the following scenario. First, the linear instability grows. If the amplitude is high enough, the nonlinearity $u\partial_x u$ tries to form shocks thus developing steep gradients which are then damped by the hyperviscous term $-\partial_x^4 u$. The problem of deriving an energy estimate for the Kuramoto–Sivashinsky equation was solved by Nicolaenko *et al.* [6] for antisymmetric solutions. The generalization to solutions without symmetry and a sharper energy estimate was provided by Collet *et al.* [7]. The trick in both approaches consisted of introducing a suitable background flow which could absorb the instability. In the next section, this technique is applied to the perturbed Hasegawa-Mima equation (2).

2. Energy estimate

Similar to the case of the Kuramoto–Sivashinsky equation, we introduce a background flow

$$\phi(x,y) = \tilde{\phi}(x,y) + \phi_0(x) \quad , \tag{9}$$

where $\phi_0(x)$ is an antisymmetric function. Equation (2) now takes the form

$$\begin{aligned} \partial_t(1-\Delta)\tilde{\phi} \quad &= -\kappa\partial_y\tilde{\phi} - \alpha\partial_{yy}\tilde{\phi} + \{\tilde{\phi},\Delta\tilde{\phi}\} + \beta\{\tilde{\phi},\partial_y\tilde{\phi}\} + \mu\Delta^3\tilde{\phi} \\ &\quad + (\partial_x\phi_0)(\partial_y\Delta\tilde{\phi}) - (\partial_y\tilde{\phi})(\partial_x^3\phi_0) + \beta(\partial_x\phi_0)(\partial_{yy}\tilde{\phi}) + \mu\Delta^3\phi_0 \quad . \end{aligned} \tag{10}$$

We multiply with $\tilde{\phi}$ and integrate over space to obtain

$$\begin{aligned} \frac{1}{2}\partial_t \int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2\right) d\Omega &= \alpha \int (\partial_y\tilde{\phi})^2 d\Omega - \mu \int \left(\nabla\Delta\tilde{\phi}\right)^2 d\Omega \\ &\quad + \int \left(\tilde{\phi}(\partial_x\phi_0)(\partial_y\Delta\tilde{\phi}) - \tilde{\phi}(\partial_y\tilde{\phi})(\partial_x^3\phi_0)\right) d\Omega \\ &\quad - \beta \int (\partial_x\phi_0)(\partial_y\tilde{\phi})^2 d\Omega - \mu \int \left(\partial_x^3\phi_0\right)\left(\partial_x^3\tilde{\phi}\right) d\Omega \quad . \end{aligned}$$

By partial integration this can be simplified to

$$\begin{aligned} \frac{1}{2}\partial_t \int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2\right) d\Omega &= \alpha \int (\partial_y\tilde{\phi})^2 d\Omega - \mu \int \left(\nabla\Delta\tilde{\phi}\right)^2 d\Omega - \int (\partial_x\phi_0)(\partial_y\tilde{\phi})(\partial_{xx}\tilde{\phi}) d\Omega \\ &\quad - \beta \int (\partial_x\phi_0)(\partial_y\tilde{\phi})^2 d\Omega - \mu \int \left(\partial_x^3\phi_0\right)\left(\partial_x^3\tilde{\phi}\right) d\Omega \quad . \end{aligned}$$

The stabilizing effect of the additional nonlinearity $\beta\{\phi,\partial_y\phi\}$ can now be understood. If ϕ_0 was a linear function in x, $\phi_0 = \frac{\alpha}{\beta}x$, then the instability could be absorbed. However, $\phi_0 = \frac{\alpha}{\beta}x$ does not fulfil the boundary condition (4) and a suitable approximation satisfying the boundary condition must be found. Therefore, a Fourier ansatz

$$\tilde{\phi}(x,y) = i \sum_{n\in\mathbf{Z}} \hat{\phi}_n(y) e^{inqx} \quad , \quad q = \frac{\pi}{L_x} \quad , \tag{11}$$

is introduced, where the Fourier-coefficients are y-dependent. Since the function is antisymmetric in x and in addition real the Fourier-coefficients obey

$$\hat{\phi}_n(y) = -\overline{\hat{\phi}}_{-n}(y) \quad , \quad \hat{\phi}_n(y) = -\hat{\phi}_{-n}(y) \in \mathbf{R} \quad , \quad \hat{\phi}_0(y) = 0 \quad . \tag{12}$$

Similarly, an ansatz for ϕ_0 is chosen

$$(\partial_x\phi_0) = -\sum_{n\in\mathbf{Z}} \hat{\psi}_n e^{inqx} \quad , \quad \hat{\psi}_n = \hat{\psi}_{-n} \in \mathbf{R} \quad , \quad \hat{\psi}_0 = 0 \quad . \tag{13}$$

Equation (11) takes now the form

$$\begin{aligned} \frac{1}{2}\partial_t \int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2\right) d\Omega &= 2\alpha L_x \int \sum_{n\in\mathbf{Z}} (\partial_y\hat{\phi}_n)^2 dy \\ &\quad - 2\mu L_x \int \sum_{n\in\mathbf{Z}} \left((nq)^6\hat{\phi}_n^2 + 3(nq)^4(\partial_y\hat{\phi}_n)^2 + 3(nq)^2(\partial_y^2\hat{\phi}_n)^2 + (\partial_y^3\hat{\phi}_n)^2\right) dy \\ &\quad + 2L_x \int \sum_{k+l+m=0} (qm)^2\hat{\psi}_k(\partial_y\hat{\phi}_l)\hat{\phi}_m dy \\ &\quad - 2\beta L_x \int \sum_{k+l+m=0} \hat{\psi}_k(\partial_y\hat{\phi}_l)(\partial_y\hat{\phi}_m) dy - \mu \int \left(\partial_x^3\phi_0\right)\left(\partial_x^3\tilde{\phi}\right) d\Omega \quad . \end{aligned} \tag{14}$$

The following simplifications can be made. First, the sum over $k+l+m=0$ is converted into a sum over k and l where m is substituted by $m=-k-l$. Next, the symmetry (13) is used to change the sum over $n \in \mathbf{Z}$

$$\sum_{n\in\mathbf{Z}} (\partial_y \hat{\phi}_n)^2 = 2\sum_{n>0} (\partial_y \hat{\phi}_n)^2 \quad .$$

The same replacement is used for the second term on the right hand side of equation (15). Finally, the sums over $k \in \mathbf{Z}$, $l \in \mathbf{Z}$ are replaced, again using the symmetries (13)

$$\begin{aligned}
\sum_{k,l} (ql)^2 (\partial_y \hat{\phi}_k) \hat{\phi}_l \, \hat{\psi}_{-k-l} &= 2 \sum_{k,l>0} (ql)^2 (\partial_y \hat{\phi}_k) \hat{\phi}_l \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{|k-l|} \right\} \quad , \\
\sum_{k,l} (\partial_y \hat{\phi}_k)(\partial_y \hat{\phi}_l) \, \hat{\psi}_{-k-l} &= 2 \sum_{k,l>0} (\partial_y \hat{\phi}_k)(\partial_y \hat{\phi}_l) \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{|k-l|} \right\} \quad .
\end{aligned}$$

Using this simplifications, equation (15) changes to

$$\begin{aligned}
\frac{1}{2}\partial_t \int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2 \right) d\Omega = {} & 4\alpha L_x \int \sum_{n>0} (\partial_y \hat{\phi}_n)^2 dy \\
& - 4\mu L_x \int \sum_{n>0} \left((nq)^6 \hat{\phi}_n^2 + 3(nq)^4 (\partial_y \hat{\phi}_n)^2 + 3(nq)^2 (\partial_y^2 \hat{\phi}_n)^2 + (\partial_y^3 \hat{\phi}_n)^2 \right) dy \\
& + 4L_x \int \sum_{k,l>0} (ql)^2 (\partial_y \hat{\phi}_k) \hat{\phi}_l \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{|k-l|} \right\} dy \\
& - 4\beta L_x \int \sum_{k,l>0} (\partial_y \hat{\phi}_k)(\partial_y \hat{\phi}_l) \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{|k-l|} \right\} dy \\
& - \mu \int \left(\partial_x^3 \phi_0 \right) \left(\partial_x^3 \tilde{\phi} \right) d\Omega \quad .
\end{aligned}$$

Now, in the sums $\sum_{k,l>0} \ldots$ the terms with $k=l$ are separated and by partial integration we obtain

$$\begin{aligned}
& \int \sum_{\substack{k,l>0 \\ k\neq l}} (ql)^2 (\partial_y \hat{\phi}_k) \phi_l \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{|k-l|} \right\} dy \\
& \quad = \int \sum_{k>l>0} (ql)^2 (\partial_y \hat{\phi}_k) \hat{\phi}_l \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \right\} dy + \int \sum_{l>k>0} (ql)^2 (\partial_y \hat{\phi}_k) \hat{\phi}_l \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{l-k} \right\} dy \\
& \quad = \int \sum_{k>l>0} (ql)^2 (\partial_y \hat{\phi}_k) \hat{\phi}_l \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \right\} dy - \int \sum_{l>k>0} (ql)^2 \hat{\phi}_k (\partial_y \hat{\phi}_l) \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{l-k} \right\} dy \\
& \quad = \int \sum_{k>l>0} q^2 (l^2 - k^2) (\partial_y \hat{\phi}_k) \hat{\phi}_l \left\{ \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \right\} dy \quad .
\end{aligned}$$

This implies

$$\frac{1}{2}\partial_t \int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2\right) d\Omega = 4L_x \int \sum_{n>0} \left[\alpha - \beta\hat{\psi}_{2n} - 3\mu(nq)^4\right] (\partial_y\hat{\phi}_n)^2$$

$$- 4\mu L_x \int \sum_{n>0} \left[(nq)^6\hat{\phi}^2 + 3(nq)^2(\partial_y^2\hat{\phi}_n)^2 + (\partial_y^3\hat{\phi}_n)^2\right] dy$$

$$+ 4L_x \int \sum_{k>l>0} q^2(l^2-k^2)(\partial_y\hat{\phi}_k)\hat{\phi}_l \left\{\hat{\psi}_{k+l} - \hat{\psi}_{k-l}\right\} dy$$

$$- 8\beta L_x \int \sum_{k>l>0} (\partial_y\hat{\phi}_k)(\partial_y\hat{\phi}_l) \left\{\hat{\psi}_{k+l} - \hat{\psi}_{k-l}\right\} dy$$

$$- \mu \int \left(\partial_x^3\phi_0\right) \left(\partial_x^3\tilde{\phi}\right) d\Omega \quad .$$

Now, choose

$$\hat{\psi}_{2n} = \frac{\alpha}{\beta} \quad \text{for} \quad n \leq \frac{1}{q}\left(\frac{\alpha}{2\mu}\right)^{\frac{1}{4}} \quad , \quad \hat{\psi}_{2n} \geq 0 \quad \text{for} \quad n > \frac{1}{q}\left(\frac{\alpha}{2\mu}\right)^{\frac{1}{4}} \quad .$$

Then it follows that the instability is suppressed and it holds

$$\alpha - \beta\hat{\psi}_{2n} - 3\mu(nq)^4 \leq -\mu(nq)^4 \quad .$$

From this we obtain

$$\frac{1}{2}\partial_t \int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2\right) d\Omega$$

$$= \; - 4\mu L_x \int \sum_{n>0} \left[(nq)^6\hat{\phi}_n^2 + (nq)^4(\partial_y\hat{\phi}_n)^2 + 3(nq)^2(\partial_y^2\hat{\phi}_n)^2 + (\partial_y^3\hat{\phi}_n)^2\right] dy$$

$$+ 4L_x \int \sum_{l=1} \sum_{k=l+1} (lq+kq)(kq)^2\hat{\phi}_k(lq)^2(\partial_y\hat{\phi}_l)\frac{lq-kq}{(kq)^2(lq)^2} \left\{\hat{\psi}_{k+l} - \hat{\psi}_{k-l}\right\} dy$$

$$- 8\beta L_x \int \sum_{l=1} (lq)^2(\partial_y\hat{\phi}_l) \sum_{k=l+1} (kq)^2(\partial_y\hat{\phi}_k)\frac{\hat{\psi}_{k+l} - \hat{\psi}_{k-l}}{(kq)^2(lq)^2} dy$$

$$- \mu \int \left(\partial_x^3\phi_0\right) \left(\partial_x^3\tilde{\phi}\right) d\Omega \quad .$$

Using the Schwarz-inequality yields

$$\frac{1}{2}\partial_t \int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2\right) d\Omega$$

$$\leq \; - 4\mu L_x \int \sum_{n>0} \left[(nq)^6\hat{\phi}_n^2 + (nq)^4(\partial_y\hat{\phi}_n)^2 + 3(nq)^2(\partial_y^2\hat{\phi}_n)^2 + (\partial_y^3\hat{\phi}_n)^2\right] dy$$

$$+ 8L_x \int \left(\sum_{n>0} (nq)^6\hat{\phi}_n^2\right)^{\frac{1}{2}} \left(\sum_{n>0} (nq)^4(\partial_y\hat{\phi}_n)^2\right)^{\frac{1}{2}}$$

$$\times \left(\sum_{k>l>0} \frac{(kq-lq)^2}{(kq)^4(lq)^4} \mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid^2\right)^{\frac{1}{2}} dy$$

$$+ 8\beta L_x \int \left(\sum_{n>0} (nq)^4(\partial_y\hat{\phi}_n)^2\right) \left(\sum_{k>l>0} \frac{\mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid^2}{(kq)^4(lq)^4}\right)^{\frac{1}{2}} dy$$

$$- \mu \int \left(\partial_x^3\phi_0\right) \left(\partial_x^3\tilde{\phi}\right) d\Omega$$

$$
\begin{aligned}
\leq\ & -4\mu L_x \int \sum_{n>0} \left[(nq)^6 \hat{\phi}^2 + (nq)^4 (\partial_y \hat{\phi}_n)^2 + 3(nq)^2 (\partial_y^2 \hat{\phi}_n)^2 + (\partial_y^3 \hat{\phi}_n)^2 \right] dy \\
& + \frac{4}{q^3} L_x \int \left(\sum_{n>0} (nq)^6 \hat{\phi}_n^2 + \sum_{n>0} (nq)^4 (\partial_y \hat{\phi}_n)^2 \right) \\
& \qquad \times \left(\sum_{k>l>0} \frac{(k-l)^2}{k^4 l^4} \mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid^2 \right)^{\frac{1}{2}} dy \\
& + 8 \frac{\beta}{q^4} L_x \int \left(\sum_{n>0} (nq)^4 (\partial_y \hat{\phi}_n)^2 \right) \left(\sum_{k>l>0} \frac{\mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid^2}{k^4 l^4} \right)^{\frac{1}{2}} dy \\
& - \mu \int \left(\partial_x^3 \phi_0 \right) \left(\partial_x^3 \tilde{\phi} \right) d\Omega \quad .
\end{aligned}
$$

Next, we want to estimate

$$
\sum_{k>l>0} \frac{(k-l)^2}{k^4 l^4} \mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid^2 \quad .
$$

For this we choose, similar to Collet *et al.* [7],

$$
\hat{\psi}_n = \begin{cases} \frac{\alpha}{\beta} & \text{for} \quad 1 \leq \mid n \mid \leq M \\ \frac{\alpha}{\beta} f(\frac{|n|}{M} - 1) & \text{for} \quad \mid n \mid > M \end{cases} \tag{15}
$$

where f is a non-increasing smooth function, which satisfies $f(0) = 1$, $f'(0) = 0$, $f \geq 0$, $\sup \mid f' \mid \leq 1$, $\int_0^\infty (1+k)^4 \mid f(k) \mid^2 dk < \infty$. This implies that $\mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid = 0$ for $k + l \leq M$ and $\mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid \leq 2 \frac{l}{M} \frac{\alpha}{\beta}$ for $k > l$. From this follows

$$
\begin{aligned}
\sum_{k>l>0} \frac{(k-l)^2}{k^4 l^4} \mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid^2 \ & \leq \ \sum_{k=\frac{M}{2}+1}^{\infty} \sum_{l=1}^{k-1} \frac{(k-l)^2}{k^4 l^4} \mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid^2 \\
& \leq \ \sum_{k=\frac{M}{2}+1}^{\infty} \sum_{l=1}^{k-1} 4 \frac{\alpha^2}{\beta^2} \frac{(k-l)^2}{k^4 l^4} \frac{l^2}{M^2} \\
& \leq \ \frac{4}{M^2} \frac{\alpha^2}{\beta^2} \sum_{k=\frac{M}{2}+1}^{\infty} \frac{1}{k^2} \sum_{l=1}^{k-1} \frac{1}{l^2} \\
& \leq \ \frac{\pi^2}{6} \frac{4}{M^2} \frac{\alpha^2}{\beta^2} \int_{\frac{M}{2}}^{\infty} \frac{1}{k^2} dk \leq \frac{2}{3} \frac{\pi^2}{M^2} \frac{\alpha^2}{\beta^2} \frac{2}{M} \leq \frac{4}{3} \frac{\alpha^2}{\beta^2} \frac{\pi^2}{M^3} \quad .
\end{aligned}
$$

Now, we choose

$$
M \geq \max \left(\frac{2}{q} \left(\frac{\alpha}{2\mu} \right)^{\frac{1}{4}}, \frac{4}{q^2} \left(\frac{\alpha^2}{\beta^2} \frac{\pi^2}{3\mu^2} \right)^{\frac{1}{3}} \right)
$$

and we obtain the estimate

$$
\sum_{k>l>0} \frac{(k-l)^2}{k^4 l^4} \mid \hat{\psi}_{k+l} - \hat{\psi}_{k-l} \mid^2 \leq \frac{\mu^2}{16} q^6 \quad . \tag{16}
$$

In a very similar way, an estimate for

$$\sum_{k>l>0} \frac{|\hat{\psi}_{k+l}-\hat{\psi}_{k-l}|^2}{k^4 l^4}$$

is derived. Again, we use the ansatz (16) where we will redefine M:

$$\sum_{k>l>0} \frac{|\hat{\psi}_{k+l}-\hat{\psi}_{k-l}|^2}{k^4 l^4} \le \sum_{k=\frac{M}{2}+1}^{\infty} \sum_{l=1}^{k-1} \frac{4l^2}{M^2}\frac{\alpha^2}{\beta^2}\frac{1}{k^4 l^4} \le \frac{4}{M^2}\frac{\alpha^2}{\beta^2} \sum_{k=\frac{M}{2}+1}^{\infty} \frac{1}{k^4} \sum_{l=1}^{k-1} \frac{1}{l^2}$$

$$\le \frac{\pi^2}{6}\frac{4}{M^2}\frac{\alpha^2}{\beta^2}\int_{\frac{M}{2}}^{\infty}\frac{1}{k^4}dk \le \frac{2}{9}\frac{\pi^2}{M^2}\frac{\alpha^2}{\beta^2}\frac{8}{M^3} = \frac{16}{9}\frac{\pi^2}{M^5}\frac{\alpha^2}{\beta^2} \ .$$

Making the choice

$$M \ge \max\left(\frac{2}{q}\left(\frac{\alpha}{2\mu}\right)^{\frac{1}{4}}, \frac{4}{q^2}\left(\frac{\alpha^2}{\beta^2}\frac{\pi^2}{3\mu^2}\right)^{\frac{1}{3}}, \left(\frac{16}{3}\frac{\pi}{q^4}\frac{\alpha}{\mu}\right)^{\frac{2}{5}}\right) \tag{17}$$

results in the estimate

$$\sum_{k>l>0} \frac{|\hat{\psi}_{k+l}-\hat{\psi}_{k-l}|^2}{k^4 l^4} < \frac{q^8}{16}\frac{\mu^2}{\beta^2} \ . \tag{18}$$

Using (17) and (19) we obtain the estimate for the time evolution of energy

$$\begin{aligned}
&\frac{1}{2}\partial_t \int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2\right) d\Omega \\
&\quad \le -\mu L_x \int \sum_{n>0} \left[3(nq)^6\hat{\phi}_n^2 + (nq)^4(\partial_y\hat{\phi}_n)^2 + 12(nq)^2(\partial_y^2\hat{\phi}_n)^2 + (\partial_y^3\hat{\phi}_n)^2\right] dy \\
&\quad\quad -\mu \int (\partial_x^3\phi_0)(\partial_x^3\tilde{\phi}) d\Omega \\
&\quad \le -\mu L_x \int \sum_{n>0} \left[3(nq)^6\hat{\phi}_n^2 + (nq)^4(\partial_y\hat{\phi}_n)^2 + 12(nq)^2(\partial_y^2\hat{\phi}_n)^2 + (\partial_y^3\hat{\phi}_n)^2\right] dy \\
&\quad\quad +\frac{\mu}{2}\int (\partial_x^3\phi_0)^2 d\Omega + \frac{\mu}{2}\int (\partial_x^3\tilde{\phi})^2 d\Omega \\
&\quad \le -\mu L_x \int \sum_{n>0} \left[(nq)^6\hat{\phi}_n^2 + (nq)^4(\partial_y\hat{\phi}_n)^2 + 12(nq)^2(\partial_y^2\hat{\phi}_n)^2 + (\partial_y^3\hat{\phi}_n)^2\right] dy \\
&\quad\quad +\frac{\mu}{2}\int (\partial_x^3\phi_0)^2 d\Omega \\
&\quad \le -\frac{\mu}{8}\min(q^6, q^4)\int \left(\tilde{\phi}^2 + (\nabla\phi)^2\right) d\Omega \\
&\quad\quad +\frac{\mu}{2}\int (\partial_x^3\phi_0)^2 d\Omega \ .
\end{aligned}$$

Here we used

$$\int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2\right) d\Omega = 4L_x \int \sum_{n>0} \left(\hat{\phi}_n^2 + (nq)^2\hat{\phi}_n^2 + (\partial_y\hat{\phi}_n)^2\right) dy \ .$$

From this immediately follows

$$\int \left(\tilde{\phi}^2 + (\nabla\tilde{\phi})^2\right) d\Omega \le \frac{4}{\min(q^6, q^4)} \int (\partial_x^3\phi_0)^2 d\Omega \ . \tag{19}$$

Now, it remains to estimate $\int(\partial_x^3\phi_0)^2 d\Omega$:

$$\int (\partial_x^3\phi_0)^2 d\Omega = 4L_xL_y\sum_{n>0}(nq)^4\hat{\psi}_n^2$$
$$\leq L_xL_y\sum_{n=1}^{M}(nq)^4\frac{\alpha^2}{\beta^2} + 4L_xL_y\int_M^\infty (nq)^4\frac{\alpha^2}{\beta^2}\mid f(\frac{n}{M}-1)\mid^2 dn$$
$$\leq C_1L_yM^5q^3\frac{\alpha^2}{\beta^2}$$

where C_1 is some numerical constant. In the last inequality the property $\int_0^\infty(1+k)^4\mid f(k)\mid^2 dk<\infty$ entered. Finally, we obtain the solution

$$\int\left(\tilde{\phi}^2+(\nabla\tilde{\phi})^2\right)d\Omega \leq C_1L_yM^5\frac{\alpha^2}{\beta^2}\frac{4}{\min(q^3,q)} \quad , \tag{20}$$

where M has to be chosen to fulfil the inequality (18).

3. Remarks

If we consider the limit $\beta \ll 1$ such that in (18) the term containing β dominates, the scaling of the energy estimate with respect to β takes the form

$$\int\left(\tilde{\phi}^2+(\nabla\tilde{\phi})^2\right)d\Omega \sim \beta^{-\frac{16}{3}} \quad .$$

If we had used the technique of Nicolaenko *et al.* [6] the energy estimate would have been proportional to $\beta^{-\frac{20}{3}}$.

Once the energy estimate is obtained, estimates for gradients are derived in exactly the same way as for the two-dimensional Navier–Stokes equation. The same is true for estimating the attractor dimension (see, e.g. Temam [8] and Constantin, Foias [9]).

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